

# Exceptionally simple exceptional models

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**ABSTRACT:** We discuss models with no dynamical vector fields in various dimensions which we claim might have exceptional symmetry on some loci of their parameter space. In particular we construct theories with four supercharges flowing to theories with global symmetry enhancing to  $F_4$ ,  $E_6$ , and  $E_7$ . The main evidence for these claims is based on extracting information about the symmetry properties of the theories from their supersymmetric partition functions.

**KEYWORDS:** Conformal Field Theory, Field Theories in Lower Dimensions

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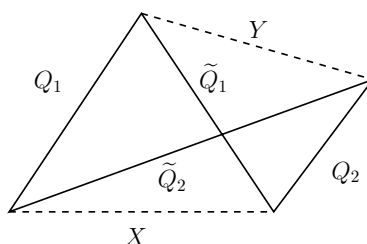
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## 1 Introduction

Some of the properties of the fixed point, IR or UV, of a general quantum field theory are not obvious from a given non-conformal description. For example, the global symmetry at the fixed point might be enhanced in dimension and/or rank. Such symmetry enhancements are often encountered when discussing gauge theories in various dimensions. For example, extended Dynkin diagram shaped  $\mathcal{N} = 2$  quiver gauge theories in three dimensions have IR fixed points with flavor Lie group corresponding to the Dynkin diagram [1]. In five dimensions  $\mathcal{N} = 1$  SU(2) gauge theories with  $N_f < 8$  flow in the UV to fixed points with  $E_{N_f+1}$  flavor symmetry [2]. The enhancement of symmetry is due to instantons in the latter case and monopoles in the former. Moreover in some cases the enhancement of symmetry might only occur on a sub locus of the conformal manifold without obvious explanation due to non-perturbative effects. As an example we mention the enhancement to  $E_7$  of the flavor symmetry of two copies of four dimensional  $\mathcal{N} = 1$  SU(2) SQCD with four flavors coupled through a quartic superpotential [3].

In this note we discuss certain models with four supercharges constructed from chiral fields and no vector fields in various dimensions. We will present evidence that, choosing the superpotentials in a careful way, these models flow to conformal theories, either in the IR or (potentially) in the UV, with conformal manifolds with possible loci having exceptional symmetries. The superpotential can be constructed in any theory allowing for four supercharges and thus will have this property in two ((2, 2) supersymmetry), three ( $\mathcal{N} = 2$  supersymmetry), or four ( $\mathcal{N} = 1$  supersymmetry) dimensions. In different dimensions the superpotential will be either relevant or irrelevant leading to fixed points with extended supersymmetry either in the IR or (possibly) UV.



**Figure 1.** The superpotential encoded in the quiver. The nodes are  $SU(2)$  flavor groups and the lines are bi-fundamental chiral fields. We have a superpotential term for each face of the tetrahedron.

The arguments in favor of enhancement of global symmetry are based on analysis of partition functions. First we show that the partition functions are invariant under the action of the Weyl group of that symmetry on the parameters, and in case these are indices can be expanded in characters of the enhanced flavor group. Moreover we will present an argument in three dimensions, generalizing the four dimensional claim [4] that in a certain order of the expansion of the index one can extract the number of marginal operators minus the currents. This will let us identify the currents of the enhanced symmetry. The physical interpretation of this result is that when the partition functions are consistent with the enhanced symmetry there is a possibility of a locus of parameter space of the theory at which the symmetry is actually enhanced.

The note is organized as follows. We will first discuss a simple example of such a model leading to  $SO(8) \times U(1) \times U(1)$ ,  $SO(10) \times U(1)$ , and  $E_6$  flavor symmetry in section two. We will proceed to deforming the superpotential to obtain a theory with  $F_4$  symmetry in section three. In section four we will consider a deformation of two copies of the theory with  $E_6$  flavor symmetry leading to a model with  $E_7$  symmetry. We will discuss several general issues following from our construction in section five.

## 2 Model with $E_6$ symmetry

Let us consider a model built from 24 chiral fields. We will organize the fields into six bi-fundamentals of  $SU(2) \times SU(2)$ . We will have four different  $SU(2)$  flavor groups and denote the chiral fields by  $Q_1$ ,  $\tilde{Q}_1$ ,  $Q_2$ ,  $\tilde{Q}_2$ ,  $X$ , and  $Y$ . The superpotential is given by,

$$W_{SO(8) \times U(1) \times U(1)} = Q_1 \tilde{Q}_1 X + Q_1 \tilde{Q}_2 Y + \tilde{Q}_2 Q_2 X + Q_2 \tilde{Q}_1 Y. \quad (2.1)$$

We can encode this superpotential in the tetrahedral quiver diagram of figure 1.

This model has the manifest symmetry  $SU(2)^4 \times U(1)_\alpha \times U(1)_t$ . The four  $SU(2)$  symmetries are manifest in the description above and under the  $U(1)_t$  the  $Q_i$  and  $\tilde{Q}_i$  have charge  $\frac{1}{2}$  while  $X$  and  $Y$  have charge  $-1$ . Under  $U(1)_\alpha$   $Q_i$  have charge 1 and  $\tilde{Q}_i$  have charge  $-1$  while  $X$  and  $Y$  have vanishing charge. All fields have R charge  $\frac{2}{3}$ . We have also summarized the various charges, including those for fields we shall introduce later in the paper, in table 1. The superpotential is irrelevant in four dimensions, and relevant in lower

	$SU(2)_1$	$SU(2)_2$	$SU(2)_3$	$SU(2)_4$	$U(1)_\alpha$	$U(1)_t$	$U(1)_R$
$Q_1$	<b>2</b>	<b>2</b>	<b>1</b>	<b>1</b>	1	$\frac{1}{2}$	$\frac{2}{3}$
$Q_2$	<b>1</b>	<b>1</b>	<b>2</b>	<b>2</b>	1	$\frac{1}{2}$	$\frac{2}{3}$
$\tilde{Q}_1$	<b>1</b>	<b>2</b>	<b>2</b>	<b>1</b>	-1	$\frac{1}{2}$	$\frac{2}{3}$
$\tilde{Q}_2$	<b>2</b>	<b>1</b>	<b>1</b>	<b>2</b>	-1	$\frac{1}{2}$	$\frac{2}{3}$
$X$	<b>2</b>	<b>1</b>	<b>2</b>	<b>1</b>	0	-1	$\frac{2}{3}$
$Y$	<b>1</b>	<b>2</b>	<b>1</b>	<b>2</b>	0	-1	$\frac{2}{3}$
$Z_+$	<b>1</b>	<b>1</b>	<b>1</b>	<b>1</b>	2	-1	$\frac{2}{3}$
$Z_-$	<b>1</b>	<b>1</b>	<b>1</b>	<b>1</b>	-2	-1	$\frac{2}{3}$
$\Lambda$	<b>1</b>	<b>1</b>	<b>1</b>	<b>1</b>	0	2	$\frac{2}{3}$

**Table 1.** The fields appearing in the models discussed in this section, together with their charges under the various global symmetries.

dimensions. Thus we will think of the model as flowing to an IR fixed point in two and three dimensions while considering a possible flow to a UV completed fixed point in four dimensions.

First, we claim that the  $SU(2)^4$  symmetry here enhances to  $SO(8)$ . We can check this by studying different supersymmetric partition functions in various dimensions. For example, in four dimensions the index is given by,<sup>1</sup>

$$1 + (\mathbf{8}_s t^{\frac{1}{2}} \alpha^{-1} + \mathbf{8}_v t^{\frac{1}{2}} \alpha + \mathbf{8}_c t^{-1})(pq)^{\frac{1}{3}} + \cdots + (4 - \mathbf{28} - 1 - 1 + \mathbf{350} + \cdots)pq + \cdots \quad (2.2)$$

The interpretation [4] of this result is that if there is a UV fixed point for which this is the index, it has an  $SO(8) \times U(1) \times U(1)$  flavor symmetry ( $-\mathbf{28} - 1 - 1$  terms in the order  $pq$  of the index corresponding to the conserved currents), and it has a conformal manifold of dimension 4 preserving this symmetry. Symmetry properties of the  $4d$  index also give the symmetry of the  $S^3$  partition function in three dimensions. In three dimensions this theory flows to a CFT in the IR. In the next section we will generalize the arguments of [4] to three dimensions, and show that also the index in three dimensions exhibits this symmetry. We can write down other partition functions in other dimensions exhibiting the symmetry (elliptic genus, spheres, indices). The details of the physics will depend on the dimension but the symmetry will remain.

We can slightly complicate the model by adding more fields. For example, we can add the fields  $Z_\pm$  which are singlets under  $SU(2)^4$ , have  $U(1)_t$  charge -1,  $U(1)_\alpha$  charges  $\pm 2$ , and R charge  $\frac{2}{3}$ . We couple these fields as,

$$W_{SO(10) \times U(1)} = W_{SO(8) \times U(1) \times U(1)} + Z_+(\epsilon \cdot \tilde{Q}_1^2 + \epsilon \cdot \tilde{Q}_2^2) + Z_-(\epsilon \cdot Q_1^2 + \epsilon \cdot Q_2^2). \quad (2.3)$$

<sup>1</sup>For notations and definitions of supersymmetric partition functions the reader can consult [5].

This theory has symmetry  $\text{SO}(10) \times \text{U}(1)_t$ , where  $\text{SO}(8) \times \text{U}(1)_\alpha$  enhances to  $\text{SO}(10)$ . Giving the example of the index in four dimensions we obtain,

$$1 + (\mathbf{10}t^{-1} + \mathbf{16}t^{\frac{1}{2}})(pq)^{\frac{1}{3}} + \cdots + (-\mathbf{45} - 1 + \overline{\mathbf{1050}} + \cdots)pq + \cdots \quad (2.4)$$

Here we deduce that we have  $\text{SO}(10) \times \text{U}(1)$  symmetry (the  $-1 - \mathbf{45}$  giving the currents) and that there is no marginal operator preserving this symmetry.

We can add another field to enhance the symmetry farther. We add a single field  $\Lambda$  which is charged under  $\text{U}(1)_t$  with charge 2 and has R charge  $\frac{2}{3}$ , and is a singlet under all the other symmetries. The superpotential is,

$$W_{E_6} = W_{\text{SO}(10) \times \text{U}(1)} + \Lambda(X^2 + Y^2 + Z_- Z_+). \quad (2.5)$$

This theory has an  $E_6$  flavor symmetry. Again in four dimensions the index is,

$$1 + \mathbf{27}(pq)^{\frac{1}{3}} + \overline{\mathbf{351}}'(pq)^{\frac{2}{3}} + (-\mathbf{78} + \mathbf{3003})pq + \cdots. \quad (2.6)$$

Like in the previous cases, this result suggests that if there is a UV fixed point, for which this is the index, it has an  $E_6$  global symmetry. One can also consider the analogue theory in three or two spacetime dimensions. Particularly, in section 2.2 we shall examine the  $3d$  index of the analogous three dimensional theory and argue that we can derive a similar result also for this case. However now the theory is expected to flow to an IR fixed point and the index can be readily interpreted as its superconformal index.

The fact that the superpotential 2.5 gives rise to  $E_6$  symmetry is not surprising. One can realize this symmetry as the group of transformations fixing the determinant of a three by three hermitian matrix built from octonions. This determinant gives the polynomial  $W_{E_6}$  with very specific numerical coefficients. Since the supersymmetric partition functions are insensitive to such parameters we allow ourselves to be agnostic about them in our discussion.

## 2.1 The moduli space

The model we presented has a moduli space spanned by the vacuum expectation values of the scalars in the chiral fields modulo the superpotential constraints. In this subsection we try to identify this space. The physics data describes it as an algebraic variety of  $\mathbb{C}^{27}$  defined by 27 quadratic equations. As a first step we note that the equations are homogeneous so the moduli space must be a complex cone over another space  $B$ .

This structure of the superpotential leads to two interesting features. First, there should be a conical singularity at the origin. This is expected as there are massless fields there. Second, there is a natural  $\text{U}(1)$  action on the cone which we identify as the  $\text{U}(1)_R$  symmetry of the theory. Indeed all fields have the same R-charge which agrees with the  $\text{U}(1)$  action on the cone.

So now we need to identify the space  $B$  which the equations define as an algebraic variety of  $\mathbb{CP}^{26}$ . We propose that this space is the complex Cayley plane which is a 16 dimensional complex manifold. This space can indeed be defined as an algebraic variety

of  $\mathbb{CP}^{26}$  via 27 quadratic equations [6]. Alternatively it can be defined as the symmetric space  $E_6/(\text{SO}(10) \times \text{U}(1))$ . This definition manifests its  $E_6$  isometry.

We can also provide additional evidence for this identification. First the Hilbert series for the complex Cayley plane was calculated in [7]. The first few terms of their results suggest the space is spanned by functions in the **27** of  $E_6$  subject to the condition that the **27** does not appear in their symmetric product. This agrees with the result we observe from the index.

We can also try to infer the dimension of the manifold from the equations. Say we choose a non-singular point on the manifold and expand the equations around this point. We can then linearize the equations and solve the resulting linear system. The dimension of the solution space is then the dimension of the manifold. Of course this only works if we choose a non-singular point. As the Cayley plane is a symmetric space, if the moduli space is as we proposed, any point save the origin will do. Say we take all fields to be zero except:  $\Lambda, (Q_1)_{22}, (Q_2)_{22}, (\tilde{Q}_1)_{22}$  and  $(\tilde{Q}_2)_{22}$  (we regard the bifundamental chirals as matrices and use the subscript as an entry in the matrix). It is easy to check that this is a solution. We then expanded around this solution and found that there is indeed a 17 dimensional solution space in accordance with our picture of the moduli space.<sup>2</sup>

## 2.2 3d supersymmetric index

We can also look at partition functions in other dimensions, notably the 3d supersymmetric index and 2d elliptic genus. We shall show that they also can be expanded in characters of  $E_6$ . We shall start with the 3d supersymmetric index as for the 3d case the  $E_6$  model leads to a conformal fixed point in the IR. For the 3d theory the supersymmetric index is given by,

$$1 + \mathbf{27}x^{\frac{2}{3}} + \overline{\mathbf{351}}'x^{\frac{4}{3}} + (-\mathbf{78} + \mathbf{3003})x^2 + \dots \quad (2.7)$$

This is very similar in structure to the 4d index. Again all states fall in characters of  $E_6$ .

An interesting question is whether we can identify the superconformal multiplets contributing to the index similarly to the results [4] we stated for the 4d index. For this we consider the possible short multiplets and their contribution to the index. The shortening conditions for various 3d superconformal algebras were extensively discussed in [8]. A more concise summary can be found in [9], and we shall employ their notations for the various short multiplets.

The 3d  $\mathcal{N}=2$  superconformal algebra contains 2 fermionic supersymmetry generators denoted as  $\mathcal{Q}$  and  $\bar{\mathcal{Q}}$ . The superconformal multiplet is then generated by acting with them, and with the translation generators  $P_\mu$ , on a superconformal primary.<sup>3</sup> Short representations are those for which the superconformal primary is annihilated by some combination of  $\mathcal{Q}$ s and  $\bar{\mathcal{Q}}$ s. Due to the superconformal algebra this necessarily fixes the dimension of the superconformal primary in term of its R-charge and angular momentum.

<sup>2</sup>Note that this requires some tuning of the constants appearing in the superpotential.

<sup>3</sup>These are states annihilated by the generators of special conformal transformations  $K_\mu$ , and their fermionic partners  $\mathcal{S}$  and  $\bar{\mathcal{S}}$ .

	Shortening conditions	Index
$A_1\bar{L}$	$\epsilon^{ab} \mathcal{Q}_a SCP\rangle_b = 0, \Delta = j - r + 1, j \geq \frac{1}{2}, r < 0$	0
$A_2\bar{L}$	$(\mathcal{Q})^2 SCP\rangle = 0, \Delta = -r + 1, j = 0, r < 0$	0
$L\bar{A}_1$	$\epsilon^{ab} \bar{\mathcal{Q}}_a SCP\rangle_b = 0, \Delta = j + r + 1, j \geq \frac{1}{2}, r > 0$	$I(2 + r + 2j, 2j + 1)$
$L\bar{A}_2$	$(\bar{\mathcal{Q}})^2 SCP\rangle = 0, \Delta = r + 1, j = 0, r > 0$	$I(2 + r, 1)$
$B_1\bar{L}$	$\mathcal{Q}_a SCP\rangle = 0, \Delta = -r, j = 0, r < -\frac{1}{2}$	0
$L\bar{B}_1$	$\bar{\mathcal{Q}}_a SCP\rangle = 0, \Delta = r, j = 0, r > \frac{1}{2}$	$I(r, 0)$
$A_1\bar{A}_1$	$\epsilon^{ab} \mathcal{Q}_a SCP\rangle_b = 0$ and $\epsilon^{ab} \bar{\mathcal{Q}}_a SCP\rangle_b = 0, \Delta = j + 1, j \geq \frac{1}{2}, r = 0$	$I(2j + 2, 2j + 1)$
$A_2\bar{A}_2$	$(\mathcal{Q})^2 SCP\rangle = 0$ and $(\bar{\mathcal{Q}})^2 SCP\rangle = 0, \Delta = 1, j = r = 0$	$I(2, 1)$
$B_1\bar{A}_2$	$\mathcal{Q}_a SCP\rangle = 0$ and $(\bar{\mathcal{Q}})^2 SCP\rangle = 0, \Delta = \frac{1}{2}, j = 0, r = -\frac{1}{2}$	$I(\frac{3}{2}, 1)$
$A_2\bar{B}_1$	$\bar{\mathcal{Q}}_a SCP\rangle = 0$ and $(\mathcal{Q})^2 SCP\rangle = 0, \Delta = \frac{1}{2}, j = 0, r = \frac{1}{2}$	$I(\frac{1}{2}, 0)$

**Table 2.** The shortening conditions and index contributions for the various short multiplets, where we have adopted the notations of [9] in the naming of the various short multiplets. We have also used  $|SCP\rangle$  for the state associated with the superconformal primary,  $\Delta$  for its conformal dimension,  $r$  for its R-charge and  $j$  for its angular momentum.

In table 2 we have summarized the various short representations, shortening conditions and their contribution to the index. For the index contribution we have defined,

$$I(l, s) = (-1)^s \frac{x^l}{1 - x^2}. \quad (2.8)$$

We now study what multiplets can contribute to the  $3d$  index at order  $x^l$  for  $l \leq 2$ . From table 2 we see that the only multiplets that can contribute are:  $A_2\bar{B}_1$ ,  $B_1\bar{A}_2$ ,  $L\bar{B}_1$  and  $A_2\bar{A}_2$ . The multiplets  $A_2\bar{B}_1$  and  $B_1\bar{A}_2$  are free fields and indeed their combination is the free chiral multiplet. The  $L\bar{B}_1$  type multiplets are chiral fields and thus their contributions are the relevant operators for  $l < 2$  and marginal operators for  $l = 2$ . The  $A_2\bar{A}_2$  is the conserved current multiplet.

From this we see that, similarly to  $4d$ , the  $x^2$  order is the marginal operators minus the conserved currents. Particularly for the  $E_6$  model we indeed see a negative contribution, at order  $x^2$ , in the adjoint of  $E_6$ . This supports our claim that this model has an IR fixed point with  $E_6$  global symmetry somewhere on its conformal manifold.

### 2.3 Other partition functions

We can also evaluate the  $2d$  elliptic genus, which is given by,

$$PE \left[ \frac{y^{\frac{1}{3}} \mathbf{27}}{1 - q} + \frac{q^{-1} y^{-\frac{1}{3}} \overline{\mathbf{27}}}{1 - q} - \frac{y^{-\frac{2}{3}} \mathbf{27}}{1 - q} - \frac{q^{-1} y^{\frac{2}{3}} \overline{\mathbf{27}}}{1 - q} \right], \quad (2.9)$$

where  $PE$  stands for plethystic exponential. The structure again has some similarities with the  $4d$  and  $3d$  indices though it contains more terms. Particularly it can be cast in characters of  $E_6$ .

We can also calculate other  $4d$  partition functions. For instance, the lens space index and the  $S^2 \times T^2$  partition function. The latter is hindered by the fact that it (see

refs. [10–12]) requires integer R-charges. We can try to correct this by mixing the  $U(1)_R$  symmetry with  $U(1)_t$  which in the  $S^2 \times T^2$  partition function formalism is associated with adding magnetic flux on  $S^2$ . Unfortunately adding the magnetic flux breaks  $E_6$  down to its  $SO(10) \times U(1)$  subgroup. The resulting partition function depends on the choices of the magnetic fluxes but can be expressed in characters of  $SO(10) \times U(1)$ .

The lens space index is quite similar to the  $4d$  index, but with more terms, and it in general depends on the chosen lens space,  $S^3/Z_k$ . A novelty in this index is that one can accommodate a non-trivial  $\mathbb{Z}_k$  holonomy on  $S^3/Z_k$  for flavor symmetries. For the model we consider without holonomies, the lens space index reads,

$$PE \left[ \frac{\mathbf{27}(pq)^{\frac{1}{3}} - \overline{\mathbf{27}}(pq)^{\frac{2}{3}}}{(1-q)(1-p)} F_k(p, q) \right]. \quad (2.10)$$

This differs from the  $4d$  index by the factor of  $F_k(p, q)$  whose exact form is given in [13]. The expression is inherently written in characters of  $E_6$ . Adding holonomies under a collection of  $U(1)$ 's will change the factor  $F_k(p, q)$  for each chiral field based on its charges under these symmetries. Naturally this will break  $E_6$ . Still we retain the action of the Weyl group which implies that different holonomies, related by the action of the Weyl group, should have the same index. This again is manifest in the expression as the chiral fields sit in characters of  $E_6$  which ensures they are properly transformed under the action of the Weyl group.

### 3 Model with $F_4$ symmetry

Let's return to the  $SO(10) \times U(1)$  model, and consider reducing the symmetry by enlarging the superpotential. Specifically, we consider breaking  $U(1)_t$  and  $U(1)_\alpha$  while preserving  $U(1)_R$  and the four  $SU(2)$  groups. Adding all terms compatible with these requirements gives the superpotential,

$$W_{F_4} = W_{SO(10) \times U(1)} + (Z_- + Z_+)(X^2 + Y^2 + \epsilon \cdot \tilde{Q}_1^2 + \epsilon \cdot \tilde{Q}_2^2 + \epsilon \cdot Q_1^2 + \epsilon \cdot Q_2^2) \\ + Z_-^3 + Z_-^2 Z_+ + Z_+^2 Z_- + Z_+^3. \quad (3.1)$$

This theory has an  $F_4$  flavor symmetry. Specifically, we consider the three dimensional model for which the  $3d$  index is,

$$1 + \mathbf{26}x^{\frac{2}{3}} + (\mathbf{324} + 1)x^{\frac{4}{3}} + (-\mathbf{52} + \mathbf{2652})x^2 + \dots. \quad (3.2)$$

We interpret this as the IR fixed point of this model having a conformal manifold with a point with enhanced symmetry which is  $F_4$  in this case. The four dimensional index, relevant for the  $4d$  model, also has a similar structure but with  $x^2$  replaced by  $pq$ . Since the superpotential is irrelevant in  $4d$ , this model is only interesting if there is a UV completed fixed point.

We can again inquire about the moduli space. The structure of the equations is quite similar so we again expect the moduli space to be a complex cone over another space  $B$ . The space  $B$  can be described as an algebraic variety of  $\mathbb{CP}^{25}$  by 26 quadratic equations.



We also expect  $B$  to have an  $F_4$  isometry. A natural guess is that  $B$  is a symmetric space similarly to the  $E_6$  case. This is reasonable as given a solution to the equations we can generate more solutions by acting with the  $F_4$  global symmetry. Assuming this covers all solutions, the resulting space is a symmetric space given by  $F_4$  moded by the symmetry keeping the solution fixed.

There are two compact symmetric spaces with  $F_4$  isometry:  $F_4/\text{SO}(9)$  (the real Cayley plane) and  $F_4/(\text{SU}(2) \times \text{USp}(6))$ . The first is 8 complex dimensional space and the second is 14 complex dimensional space. We next analyze the equations linearized around the solution where the only non-vanishing fields are:  $(Q_1)_{22}, (Q_2)_{22}, (\tilde{Q}_1)_{22}$  and  $(\tilde{Q}_2)_{22}$ . We find a 15 dimensional solution space.<sup>4</sup> This is consistent with the moduli space being a complex cone over the symmetric space  $F_4/(\text{SU}(2) \times \text{USp}(6))$ .

#### 4 Model with $E_7$ symmetry

We can use the  $E_6$  model to generate a model with  $E_7$  global symmetry. To do this we take 56 chiral multiplets and split them into two copies of the 27 chiral fields in the  $E_6$  model and two additional chiral fields  $P_+$  and  $P_-$ . The fields interact through the superpotential,

$$W_{E_7} = P_+ W_{E_6}^1 + P_- W_{E_6}^2 + W_{\text{int}}, \quad (4.1)$$

where we use  $W_{E_6}^1$  and  $W_{E_6}^2$  for the superpotential of the  $E_6$  model involving chiral fields from just one of the two copies, these being the first or second copy respectively. We use  $W_{\text{int}}$  for the most general quartic superpotential involving only the combinations  $P_+ P_-$  and products of fields in the first copy with its image in the second copy.

The classical flavor symmetry is  $\text{SU}(2)^4 \times \text{U}(1)_t \times \text{U}(1)_\alpha \times \text{U}(1)_p$ . Fields belonging to one copy of the 27 transform as before under  $\text{SU}(2)^4 \times \text{U}(1)_t \times \text{U}(1)_\alpha$  while the other copy transforms as the complex conjugate. Under  $\text{U}(1)_p$  copy one has charge  $-1$  while copy two has charge  $1$ . The fields  $P_+$  and  $P_-$  are singlets under  $\text{SU}(2)^4 \times \text{U}(1)_t \times \text{U}(1)_\alpha$  and carry charge  $3$  and  $-3$  under  $\text{U}(1)_p$ , respectively.

The theory also has a  $\text{U}(1)_R$  symmetry where now the R-charge of all the fields is  $\frac{1}{2}$ . The superpotential is now quartic so it is irrelevant in four dimensions, marginally irrelevant in three dimensions and relevant in two dimensions.

We claim that this theory has  $E_7$  global symmetry. Again in four dimensions the index is,

$$\begin{aligned} &1 + \mathbf{56}(pq)^{\frac{1}{4}} + (\mathbf{1463} + \mathbf{133})(pq)^{\frac{1}{2}} + (\mathbf{24320} + \mathbf{6480})(pq)^{\frac{3}{4}} \\ &+ (-\mathbf{133} + \mathbf{293930} + \mathbf{150822} + \mathbf{7371})pq + \dots \end{aligned} \quad (4.2)$$

The three dimensional index also has a similar structure but with  $pq$  replaced by  $x^2$ . If either the  $3d$  or the  $4d$  models possess a UV completed fixed point, then the indices suggest it should have an  $E_7$  global symmetry.

The analogue two dimensional model is expected to flow to an IR fixed point, the elliptic genus of which, can be cast in characters of  $E_7$ . Therefore one may also expect this IR fixed point to have an  $E_7$  global symmetry at a point on its conformal manifold.

<sup>4</sup>This requires some tuning of the constants in the superpotential.

## 5 General properties

Finally, we wish to discuss some general properties that emerge from our construction. Specifically we seek to summarize the salient features of our construction in a way that facilitates generalizations to other systems. In general we have a collection of chiral fields that we choose to form a representation  $R$  of a chosen group  $G$ , where for simplicity we consider only a single representation of  $G$ . The chiral fields carry charges under the classical symmetry so that they correctly form the representation  $R$  of  $G$ .

This generally requires a superpotential to force all fields to carry the desired charges and eliminates additional symmetries. We shall limit ourselves to theories with an  $R$ -symmetry as in these cases we can perform more stringent tests using the superconformal index. Furthermore, as by assumption all chiral fields form a single representation  $R$  of  $G$ , they must have the same  $R$ -charge. The results of these two conditions is that the superpotential must be a polynomial in the fields of degree  $r$ . One obstruction for this construction is that one must be able to find the desired superpotential. We can formulate some necessary conditions using group theory.

First, group theory gives a limitation on the possible values of  $r$ . The chiral ring of the theory is made from the symmetric products of the chiral fields and so is in  $G$  representations appearing in such products. The superpotential constraints eliminate chiral ring elements made from the  $r-1$  symmetric product of the basic chiral fields, and carry charges in the conjugate representation to  $R$ . Thus consistency necessitates that the representation  $\bar{R}$  must appear in  $\otimes_{Sym}^{r-1} R$ . This in turn constrains  $r$ .

For example, for  $E_6$  we have  $\otimes_{Sym}^2 \mathbf{27} = \mathbf{\overline{351}'} + \mathbf{\overline{27}}$  so the minimal possible value of  $r$  is 3. Likewise for  $F_4$ ,  $\otimes_{Sym}^2 \mathbf{26} = \mathbf{324} + \mathbf{26} + \mathbf{1}$  so the minimal non-trivial value of  $r$  is again 3. However for  $E_7$  we have  $\otimes_{Sym}^2 \mathbf{56} = \mathbf{1463} + \mathbf{133}$  so a cubic superpotential is not possible. Yet  $\otimes_{Sym}^3 \mathbf{56} \supset \mathbf{56}$  so the minimal non-trivial value of  $r$  in this case is 4.

An additional condition can then be given using the  $4d$  supersymmetric index (or as we have seen the three dimensional one). This receives contributions from the chiral fields modulo the superpotential constraints. A nice feature of the  $4d$  index is that the  $pq$  order receives contributions only from marginal operators, which contribute positively, and conserved currents, which contribute negatively. Therefore we can look at the negative terms in the  $pq$  order and see whether or not we indeed get the adjoint, and only the adjoint representation of  $G$ . In fact it is straightforward to write the contribution for the  $pq$  order to be:  $\otimes_{Sym}^r R - R \otimes \bar{R}$ . This essentially reduces the problem to group theory: which representation appearing in the direct product  $R \otimes \bar{R}$  do not appear in  $\otimes_{Sym}^r R$ . For example, in the  $E_6$ ,  $F_4$  and  $E_7$  theories the answer to this is indeed only the adjoint representation.

### 5.1 Example: $G_2$

As an illustrating example let's consider the exceptional group  $G_2$  and its  $\mathbf{7}$  dimensional representation. It is convenient to form the chiral fields in representations of the  $SU(3)$  maximal subgroup of  $G_2$ . Under it the  $\mathbf{7}$  of  $G_2$  decomposes as  $\mathbf{1} + \mathbf{3} + \mathbf{\bar{3}}$  so we shall use 3 chiral fields  $F$  in the  $\mathbf{3}$  of the classical  $SU(3)$ , 3 chiral fields  $\bar{F}$  in the  $\mathbf{\bar{3}}$  and a singlet  $X$ .

Next we need to find a superpotential that limits the fields to these charges. However we shall now argue that this is not possible.

The superpotential must be  $SU(3)$  invariant and so must be made from the meson  $F\bar{F}$ . Note that baryonic products vanish as the fields are bosonic. This implies that the minimal non-trivial order for the superpotential is 4, and also that there is an additional  $U(1)$  under which  $F$  and  $\bar{F}$  carry opposite charges that we cannot eliminate. The superpotential that we can add has the form,

$$W = (F\bar{F})^2 + F\bar{F}X^2 + X^4. \quad (5.1)$$

This leads to a classical  $U(1) \times SU(3)$  global symmetry under which the fields are charged as:  $\mathbf{1}^0 + \mathbf{3}^1 + \bar{\mathbf{3}}^{-1}$ . These in fact form the  $\mathbf{7}$  of  $SO(7)$  under its  $U(1) \times SU(3)$  subgroup. So we conclude that we cannot build a  $G_2$  model. Attempting to build one leads to model with  $SO(7)$  global symmetry.

We can also see all these statements materialize just from group theory analysis. First note that  $\otimes_{Sym}^2 \mathbf{7} = \mathbf{27} + \mathbf{1}$  and  $\otimes_{Sym}^3 \mathbf{7} = \mathbf{77} + \mathbf{7}$  so indeed the minimal non-trivial order of the superpotential is 4. Next we look at the conserved currents given by the terms in the product  $\mathbf{7} \otimes \mathbf{7}$  that are not contained in  $\otimes_{Sym}^4 \mathbf{7}$ . Doing the group theory we find these to be the adjoint  $\mathbf{14}$  of  $G_2$  and the  $\mathbf{7}$ . Thus we see that there are additional conserved currents, which in fact form the adjoint of  $SO(7)$  signaling that such a theory must have a larger global symmetry. So the group theory analysis supports the previous claim that there is no analogous model with  $G_2$  as its global symmetry.

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